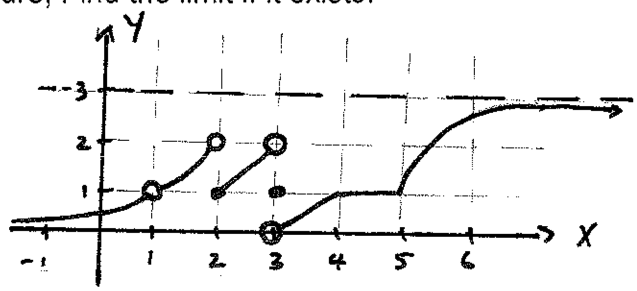
CHAPTER 1 REVIEW EXERCISES

1. For the function f graphed in the accompanying figure, Find the limit if it exists.
   Use the figure given question 1 in your textbook



   a) $\lim_{x \to 1} f(x)$

Solution:

If $\lim_{x\to 1} f(x)$ is to exist,

then $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

Our goal is to describe the behavior of f "near" $x=1$.
Notice that the function f is not defined at $x=1$.
yet, if $x \to 1$ and $x \neq 1$, then $f(x) \to 1$.
So, the number 1 is crucial in describing the behavior of f near $x=1$.
We say that 1 is the limit of $f(x)$ as x approaches 1.
This is written compactly as $\lim_{x\to 1} f(x) = 1$

   b) $\lim_{x \to 2} f(x)$ DNE

$\lim_{x\to 2^-} f(x) = 2$

$\lim_{x\to 2^+} f(x) = 1$

   c) $\lim_{x \to 3} f(x)$ Does not exist "DNE"

$\lim_{x\to 3^-} f(x) = 2$; $\lim_{x\to 3^+} f(x) = 0$

$\therefore \lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

   d) $\lim_{x \to 4} f(x) = 1$

$\lim_{x\to 4^-} f(x) = 1$

$\lim_{x\to 4^+} f(x) = 1$

   e) $\lim_{x \to +\infty} f(x) = 3$

   F) $\lim_{x \to -\infty} f(x) = 0$

   g) $\lim_{x \to 3^+} f(x) = 0$

   h) $\lim_{x \to 3^-} f(x) = 2$

   i) $\lim_{x \to 0} f(x) = \frac{1}{2}$

2. In each part, complete the table and make a conjecture about the value of the limit indicated. Confirm your conjecture by finding the limit analytically.

a) $f(x)=\frac{x-2}{x^2-4}$; $\lim_{x\to 2^+} f(x)$

Use Calculator.

x	2.00001	2.0001	2.001	2.01	2.1	2.5
$f(x)$						

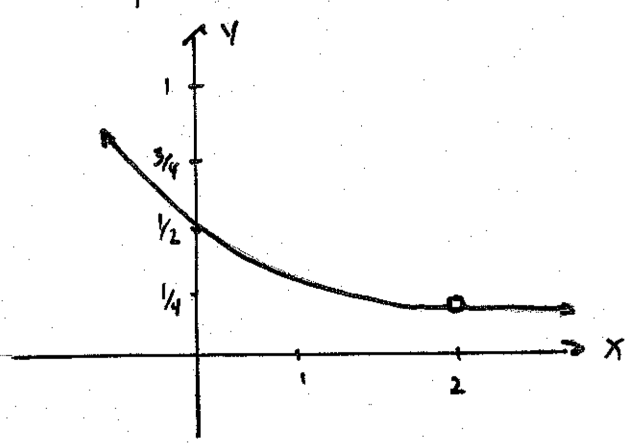
From the table limit appears to be $\simeq 0.25$

→ Begin by thinking about the function $f(x)=\frac{x-2}{x^2-4}$.

Domain: All $x \in \mathbb{R}$ Except ± 2.

For $x \neq 2$, $f(x)=\frac{x-2}{x^2-4}=\frac{\cancel{x-2}}{(\cancel{x-2})(x+2)}=\frac{1}{x+2}$.

Graph of f:



Note: The function f is not defined at $x=2$, but the value of $f(x)$ can be made as close as we like to $1/4$ simply by choosing x sufficiently close, but not equal to 2.

Hence $\lim_{x\to 2^+}\frac{x-2}{x^2-4}=\frac{0}{0}$ factor

$\lim_{x\to 2^+}\frac{\cancel{x-2}}{(\cancel{x-2})(x+2)}=\lim_{x\to 2^+}\frac{1}{x+2}\to\frac{1}{4}$

∴ Our conjecture is right.

b) $f(x)=\frac{\tan 4x}{x}$; $\lim_{x\to 0} f(x)$

Calculator

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	4.0021347	4.0000213	4.0000002	4.0000002	4.0000213	4.0021347

From the table limit appears to be close to 4.

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\tan 4x}{x}$$

$$=\lim_{x\to 0}\frac{\sin 4x}{x\cos 4x}$$

$$=\lim_{x\to 0}\frac{\sin 4x}{x}\cdot\frac{1}{\cos 4x}$$

$$=\lim_{x\to 0} 4\left(\frac{\sin 4x}{4x}\right)\cdot\frac{1}{\cos 4x}=4(1)\left(\frac{1}{1}\right)=4.$$

Note:

$\lim_{x\to 0}\frac{\sin x}{x}=1$

and

$\tan x=\frac{\sin x}{\cos x}$

Hence Our conjecture is right.

3. a) Approximate the value for the limit

$$\lim_{x\to 0}\frac{3^x-2^x}{x}$$

$f(x) = \frac{3^x - 2^x}{x}$

Domain: All $x \in \mathbb{R}$ Except 0.

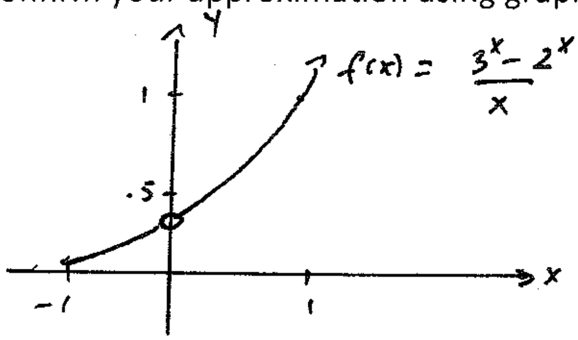
to three decimal places by constructing an appropriate table of values.

Observe: $f(x) = \frac{3^x - 2^x}{x}$, $x \to 0$ and $x \neq 0$.

x	-.1	-.01	-.001	-.0001	-.00001	0	.00001	.0001	.001	.01	.1
f(x)	.371	.402	.405	.405	.405	undefined	.405	.406	.406	.409	.443

$\Rightarrow$ limit appears to be .405 $\quad \therefore \lim_{x\to 0}\frac{3^x-2^x}{x} \approx 0.405$

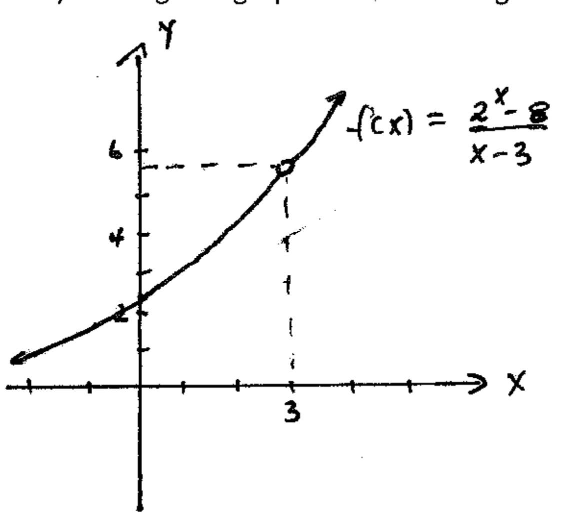
c) Confirm your approximation using graphical evidence.



Hence from the graph $\lim_{x\to 0}\frac{3^x-2^x}{x} \approx 0.405$

4) *Approximate* $\lim_{x\to 3}\frac{2^x-8}{x-3}$

both by looking at a graph and calculating values for some appropriate choices of x.



Choose x very close to 3, but not equal to 3.

x	f(x)
3.0001	5.545
3.00001	5.545
3	undefined
2.9999	5.545
2.999	5.543

$\lim_{x\to 3}\frac{2^x-8}{x-3} \approx 5.545$ //

5-10 Find the limits.

5) $\lim\limits_{x\to -1}\dfrac{x^3-x^2}{x-1}$

Solution

plug in $x=-1$

$$\lim_{x\to -1}\frac{x^3-x^2}{x-1}=\frac{-1-1}{-1-1}=\frac{-2}{-2}=1$$

$$\therefore \lim_{x\to -1}\frac{x^3-x^2}{x-1}=1$$

6) $\lim\limits_{x\to 1}\dfrac{x^3-x^2}{x-1}$

Solution

plug in $x=1$

$$\lim_{x\to 1}\frac{x^3-x^2}{x-1}=\frac{0}{0}\quad \text{factor}$$

$$\lim_{x\to 1}\frac{x^2\cancel{(x-1)}}{\cancel{x-1}}$$

$$\lim_{x\to 1} x^2=1$$

7) $\lim\limits_{x\to -3}\dfrac{3x+9}{x^2+4x+3}$

Solution

Plug in $x=-3$

$$\lim_{x\to -3}\frac{3x+9}{x^2+4x+3}=\frac{-9+9}{9-12+3}=\frac{0}{0}\quad \text{Factor}$$

$$\lim_{x\to -3}\frac{3\cancel{(x+3)}}{\cancel{(x+3)}(x+1)}$$

$$\lim_{x\to -3}\frac{3}{x+1}=\frac{3}{-2}$$

8) $\lim\limits_{x\to 2^-}\dfrac{x+2}{x-2}$

solution

Plug in $x=2$

$$\lim_{x\to 2^-}\frac{x+2}{x-2}=\frac{4}{0}\quad \frac{k}{0}$$

Test 1.999 2

$$\therefore \lim_{x\to 2^-}\frac{x+2}{x-2}=-\infty$$

$$f(x)=\frac{x+2}{x-2}=\frac{+}{-}=-\infty$$

9) $\lim\limits_{x\to +\infty}\dfrac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)}$

Solution

Divide the numerator and denominator by the highest power of x. in this case x^5.

$$\lim_{x\to +\infty}\frac{\left[x\left(2-\frac{1}{x}\right)\right]^5}{\left[x^2\left(3+\frac{2}{x}-\frac{7}{x^2}\right)\left(x^3\left(1-\frac{9}{x^2}\right)\right)\right]}$$

$$\lim_{x\to +\infty}\frac{\cancel{x^5}\left(2-\frac{1}{x}\right)^5}{\cancel{x^5}\left(3+\frac{2}{x}-\frac{7}{x^2}\right)\left(1-9/x^2\right)}=\frac{2^5}{(3)(1)}=32/3$$

10) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}=0/0$

$$\lim_{x\to 0}\frac{\left(\sqrt{x^2+4}-2\right)\left(\sqrt{x^2+4}+2\right)}{x^2\left(\sqrt{x^2+4}+2\right)}$$

$$\lim_{x\to 0}\frac{x^2+\cancel{4}-\cancel{4}}{x^2\left(\sqrt{x^2+4}+2\right)}$$

$$\lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)}$$

$$\lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}=\frac{1}{4}$$

11. **In each part, find the horizontal asymptotes, if any**

a) $y = \dfrac{2x-7}{x^2-4x}$

<u>Calculus:</u> Calculate the limit.

$$\lim_{x\to\infty} \frac{2x-7}{x^2-4x}$$

Divide the numerator and the denominator by the highest power of x. In this case, x^2.

$$\lim_{x\to\infty} \frac{x\left(2-\frac{7}{x}\right)}{x^2\left(1-\frac{4}{x}\right)}$$

$$\lim_{x\to\infty} \frac{2-\overbrace{\frac{7}{x}}^{=0}}{x\left(1-\underbrace{\frac{4}{x}}_{=0}\right)}$$

Note: Basic fact of limit.

$$\lim_{x\to\infty} \frac{k}{x^n} = 0$$

$$\lim_{x\to\infty} \frac{2}{x} = 0$$

Hence $y=0$ is the horizontal asymptote of y.

b) $y = \dfrac{x^3-x^2+10}{3x^2-4x}$

$$\lim_{x\to\infty} \frac{x^3-x^2+10}{3x^2-4x}$$

$$\lim_{x\to\infty} \frac{\cancel{x^3}\left(1-\overbrace{\frac{1}{x}}^{=0}+\overbrace{\frac{10}{x^3}}^{=0}\right)}{\cancel{x^2}\left(3-\underbrace{4/x}_{=0}\right)}$$

$$\lim_{x\to\infty} \frac{x(1)}{3} \Rightarrow \lim_{x\to\infty} \frac{x}{3} = \infty$$

no horizontal asympote.

d) $y = \dfrac{2x^2-6}{x^2+5x}$

$$\lim_{x\to\infty} \frac{2x^2-6}{x^2+5x}$$

$$\lim_{x\to\infty} \frac{\cancel{x^2}\left(2-6/x^2\right)}{\cancel{x^2}\left(1+5/x\right)}$$

$$\lim_{x\to\infty} \frac{2-6/x^2}{1+5/x} = \frac{2}{1} = 2$$

Horizontal asympote is 2.

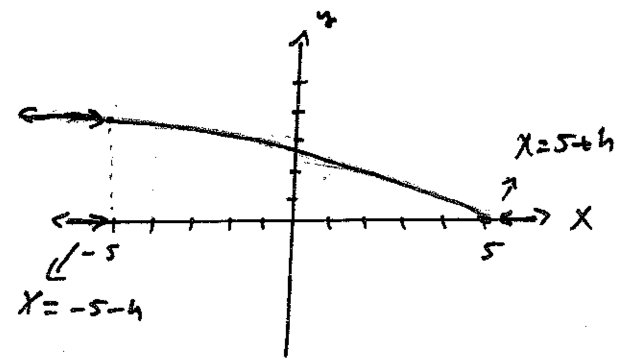
12. In each part, find $\lim_{x \to a} f(x)$, if it exists,

where a is replaced by $0, 5^+, -5^-, 5, -\infty, +\infty$.

a) $f(x) = \sqrt{5-x}$

Graph of f:

Domain: $\{ x \mid x \leq 5 \}$



- $\lim_{x \to 0} \sqrt{5-x} = \sqrt{5}$
- $\lim_{x \to 5^+} \sqrt{5-x} = \lim_{h \to 0} \sqrt{5-(5+h)}$
  $= \lim_{h \to 0} \sqrt{5-5-h}$
  $= \lim_{h \to 0} \sqrt{-h} \Rightarrow$ limit does not exist
- $\lim_{x \to -\infty} \sqrt{5-x} = +\infty$ see graph
- $\lim_{x \to -5^-} \sqrt{5-x} = \lim_{h \to 0} \sqrt{5-(-5-h)}$
  $= \lim_{h \to 0} \sqrt{10+h} = \sqrt{10}$
- $\lim_{x \to \infty} \sqrt{5-x} \Rightarrow$ limit does not exist
- $\lim_{x \to 5} \sqrt{5-x}$ does not exist "see the graph"

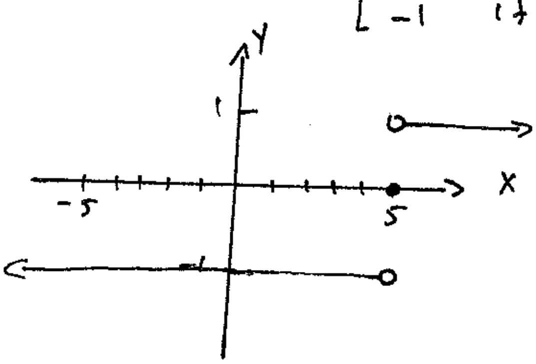
b) b) $f(x) = \begin{cases} (x-5)/|x-5|, x \neq 5 \\ 0, x = 5 \end{cases}$

Graph of $f(x) = \dfrac{x-5}{|x-5|}$

$f(x) = \dfrac{x-5}{|x-5|} = \begin{cases} \dfrac{x-5}{x-5} & \text{if } x-5>0 \\ \dfrac{x-5}{-(x-5)} & \text{if } x-5<0 \end{cases}$

$= \begin{cases} 1 & \text{if } x>5 \\ -1 & \text{if } x<5 \end{cases}$

- $\lim_{x \to 0} \dfrac{x-5}{|x-5|} = -1$ see "Graph"
- $\lim_{x \to 5^+} \dfrac{x-5}{|x-5|} = 1$ see "graph"
- $\lim_{x \to -5^-} \dfrac{x-5}{|x-5|} = -1$ see "graph"
- $\lim_{x \to 5} f(x)$ does not exist "Jump discontinuity"



- $\lim_{x \to -\infty} f(x) = -\infty$
- $\lim_{x \to +\infty} f(x) = +\infty$

13-20 Find the limits.

13. $\lim_{x\to 0}\frac{\sin 3x}{\tan 3x}$

$\lim_{x\to 0} \sin 3x \cdot \frac{1}{\tan 3x}$

$\lim_{x\to 0} \sin 3x \cdot \cot 3x$

$\lim_{x\to 0} \cancel{\sin 3x} \cdot \frac{\cos 3x}{\cancel{\sin 3x}}$

$\lim_{x\to 0} \cos 3x = \underline{1}$

14. $\lim_{x\to 0}\frac{x\sin x}{1-\cos x} = \frac{0}{0}$

$\lim_{x\to 0} \frac{x\sin x\,(1+\cos x)}{(1-\cos x)(1+\cos x)}$

$\lim_{x\to 0} \frac{x\sin x\,(1+\cos x)}{1-\cos^2 x}$

$\lim_{x\to 0} \frac{x\,\cancel{\sin x}\,(1+\cos x)}{\sin^2 x}$

$\lim_{x\to 0} \frac{x}{\sin x}(1+\cos x)$

$= 1\,(1+1)$

$= 2$

15. $\lim_{x\to 0}\frac{3x-\sin(kx)}{x}, k\neq 0$

$\lim_{x\to 0}\left[\frac{3x}{x} - \frac{\sin(kx)}{x}\right]$

$\lim_{x\to 0}\left[3 - \frac{\sin(kx)}{x}\right]$

$\lim_{x\to 0} 3 - \lim_{x\to 0}\frac{\sin(kx)}{x}\cdot\frac{k}{k}$

$\lim_{x\to 0} 3 - \lim_{x\to 0} k\left(\frac{\sin(kx)}{kx}\right)$

$3 - k(1)$

$\underline{\underline{3-k}}$

16. $\lim_{\theta\to 0}\tan\left(\frac{1-\cos\theta}{\theta}\right)$

$\tan\left[\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \frac{0}{0}\right]$

$\tan\left[\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\right]$

$\tan\left[\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\right]$

$\tan\left[\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\right]$

$\tan\left[\lim_{\theta\to 0}\sin\theta\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{1+\cos\theta}\right]$

$\tan\left[0\cdot 1\cdot\frac{1}{2}\right]$

$\tan(0)$

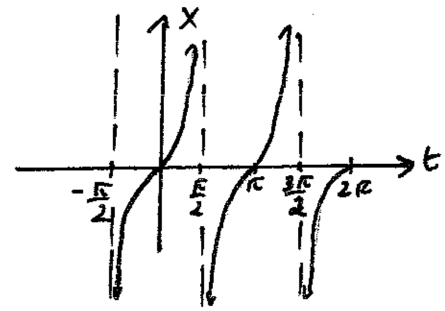
0

17. $\lim_{t \to \pi/2^+} e^{\tan t}$

Solution:

let $x = \tan t$ and See the graph of $x = \tan t$

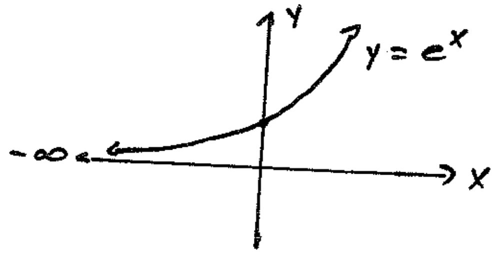
if $t \to \frac{\pi}{2}^+$, $x \to -\infty$



By Substitution:

$\lim_{x \to -\infty} e^{x} = 0$ see the graph of $f(x) = e^x$

$\therefore \lim_{t \to \frac{\pi}{2}^+} e^{\tan t} = 0$



18. $\lim_{\theta \to 0^+} \ln(\sin 2\theta) - \ln(\tan\theta)$

Solution

$\lim_{\theta \to 0^+} \ln\left(\frac{\sin 2\theta}{\tan\theta}\right)$ → Since: $\ln\left(\frac{A}{B}\right) = \ln(A) - \ln(B)$

$\lim_{\theta \to 0^+} \ln\left(\frac{2\sin\theta\cos\theta}{\sin\theta/\cos\theta}\right)$ → Since: $\sin 2\theta = 2\sin\theta\cos\theta$
$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\lim_{\theta \to 0^+} \ln\left(\frac{2\cancel{\sin\theta}\cos\theta}{1} \cdot \frac{\cos\theta}{\cancel{\sin\theta}}\right)$

$\lim_{\theta \to 0^+} \ln\left(2\cos^2(\theta)\right)$

$\lim_{\theta \to 0^+} \ln(2) + \ln\left((\cos\theta)^2\right)$ → Since: $\ln(AB) = \ln(A) + \ln(B)$

$\lim_{\theta \to 0^+} \ln 2 + 2\ln((\cos\theta)) = \ln(2) + 2\ln(1)$

$= \ln 2 + 2(0)$ → since $\ln(1) = 0$

$= \underline{\underline{\ln 2}}$

19. $\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^{-x}$

Note:

The definition of the number e is

$$\lim_{x\to a}\left(1+\frac{k}{x}\right)^{x}=e^{k}$$

for k = Real number

Solution

Rewrite to match the definition of "e"

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{-x}=\lim_{x\to\infty}\left[\left(1+\frac{3}{x}\right)^{x}\right]^{-1}$$

In this problem $k=3$

$$=\left[\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}\right]^{-1}$$

$$=\left[e^{3}\right]^{-1}=e^{-3}$$

20. $\lim_{x\to+\infty}\left(1+\frac{a}{x}\right)^{bx}, a,b>0$

Solution

Rewrite to match the definition of "e"

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$$

$$\lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^{x}\right]^{b}=\left[e^{a}\right]^{b}=e^{ab}$$

Here $k=a$

21. If $\$1000$ is invested in an account that pays 7% interest compounded n times each year, then in 10 years there will be $1000(1+0.07/n)^{10n}$ dollars in the account. How much money will be in the account in 10 years if the interest is compounded quarterly a) $(n=4)$? b) Monthly $(n=12)$?

c) Daily $(n=365)$? Determine the amount of money that will be in the account in 10 years if the interest is compounded continuously, that is, as $n \to \infty$.

Solution

a) For $n = 4$

$p = \$1000$

$r = 7\% = 0.07$

$t = 10$ yrs.

Use Formula

$$A = p\left(1+\frac{r}{n}\right)^{nt} \Rightarrow 1000\left(1+\frac{0.07}{4}\right)^{4(10)} \simeq \$\,2001.60$$

b) For $n = 12$

$$A = 1000\left(1+\frac{0.07}{12}\right)^{12(10)} \simeq \$\,2009.66$$

c) For $n = 365$

$$A = 1000\left(1+\frac{0.07}{365}\right)^{365(10)} \simeq \$\,2013.62$$

d) For $n \Rightarrow \infty$

$$\lim_{n\to\infty} 1000\left(1+\frac{0.07}{n}\right)^{10n} \rightarrow$$

$$\lim_{n\to\infty} 1000\left[\left(1+\frac{0.07}{n}\right)^{n}\right]^{10}$$

$$1000 \lim_{n\to\infty}\left[\left(1+\frac{0.07}{n}\right)^{n}\right]^{10}$$

$$1000\left(e^{0.07}\right)^{10} = 1000\,e^{0.7}$$

$$= \$2013.75$$

Note:

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x} = e^{k}$$

For $k \in \mathbb{R}$

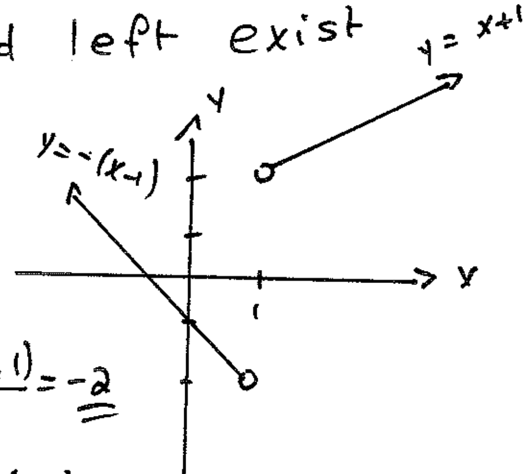
Re-writ to match this Form.

22. a) Write a paragraph or two that describes how the limit of a function can fail to exist at $x = a$, and accompany your description with some specific examples.

<u>Answer</u>

The limit of a function $f(x)$ may fail to exist if either

1) The limits from right and left exist but not equal.



Example: see "graph"

$$\lim_{x \to 1} \frac{x^2 - 1}{|x-1|}$$

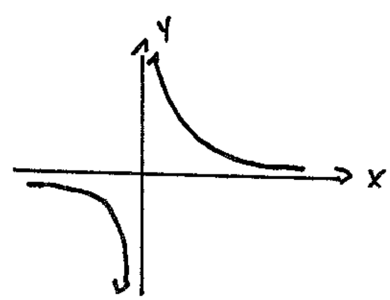
- $\displaystyle \lim_{x \to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x \to 1^-} \frac{\cancel{(x-1)}(x+1)}{-\cancel{(x-1)}} = \lim_{x \to 1^-} \frac{(x+1)}{-1} = -2$
- $\displaystyle \lim_{x \to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x \to 1^+} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x \to 1^+} \frac{(x+1)}{1} = 2$

Therefore, the limit does not exist.

2) The value of $f(x)$ may get larger and larger (tend to infinity) as $x \to a$ from both sides.

Example: "graph"

$$\lim_{x \to 0} \frac{1}{x}$$



- $\displaystyle \lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\displaystyle \lim_{x \to 0^+} \frac{1}{x} = +\infty$

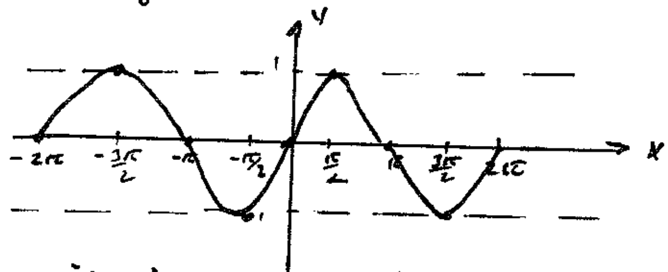
Therefore, the limit does not exist.

b) Write a paragraph or two that describes how the limit of a function can fail to exist as $x \to +\infty$ or $x \to -\infty$, and accompany your description with some specific examples.

Answer :

The limit of a function fails to exist as $x \to -\infty$ or $x \to +\infty$, when the value of $f(x)$ oscillate infinitely often, approaching no limit.

Example : $\lim_{x \to \infty} \sin x$ $\xrightarrow{\text{"graph"}}$



$\therefore \lim_{x \to \infty} \sin x = \text{DNE}$, since it is oscillating between -1 and $+1$ as $x \to \pm\infty$

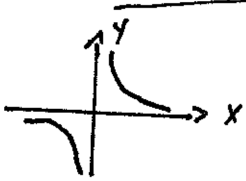
c) Write a paragraph or two that describes how a function can fail to be continuous at $x = a$, and accompany your description with some specific examples. Answer: A general definition : A function $f(x)$ is said to be continuous at $x = a$ if

1) $f(x)$ is defined at $x = a$ (I.E $f(a)$ Exist)
2) $\lim_{x \to a} f(x)$ Exists
3) $\lim_{x \to a} f(x) = f(a)$

Hence , a function can fail to be continuous if any of the above three conditions is not satisfied.

Example :

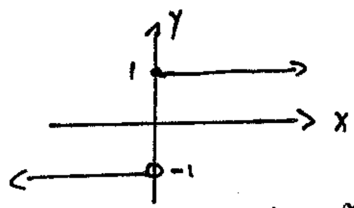
1)



$f(x) = \frac{1}{x}$

$f(0)$ not defined

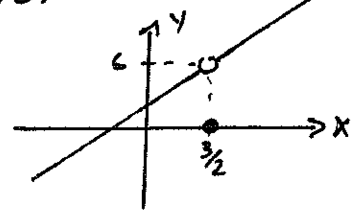
2)



$f(x) = |x| = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

$f(0)$ Defined, but $\lim_{x \to 0} f(x)$ DNE

3)



$f(x) = \begin{cases} 2x+3, & x \neq 3/2 \\ 0, & x = 3/2 \end{cases}$

1) $f(3/2) = 0$ Defined
2) $\lim_{x \to 3/2} f(x) = 6$ Exist
3) $\lim_{x \to 3/2} f(x) \neq f(3/2)$